Complex Q's before midterm

Stereographic Projection

$$\phi^{-1}(z) = (a, b, c) \in S^2 \subseteq \mathbb{R}^3$$

$$\phi^{-1}(z) = \left(\frac{2\operatorname{Re}(z)}{|z|^2+1}, \frac{2\operatorname{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$$

Example

Let P be the plane given by $x+y=z$ in $\mathbb{R}^3$. Find the image $\phi(P \cap S^2)$ in the complex plane, where ϕ is the Stereographic Projection.

$$P = \{ (x, y, c) : x+y=c \} \subseteq \mathbb{R}^3$$

$$P \cap S^2 = \{ (x, y, c) : x+y=c \text{ and } x^2+y^2+c^2=1 \}.$$

$$\phi(P \cap S^2) = \{ \phi(x, y, c) : x+y=c \text{ and } x^2+y^2+c^2=1 \}.$$

$$= \left\{ z : x = \frac{2\operatorname{Re}(z)}{|z|^2+1}, y = \frac{2\operatorname{Im}(z)}{|z|^2+1}, c = \frac{|z|^2-1}{|z|^2+1}, x+y=c, \text{ and } x^2+y^2+c^2=1 \right\}.$$

$$= \left\{ z : \frac{2\operatorname{Re}(z)}{|z|^2+1} + \frac{2\operatorname{Im}(z)}{|z|^2+1} = \frac{|z|^2-1}{|z|^2+1} \text{ and } \left[\frac{2\operatorname{Re}(z)}{|z|^2+1}\right]^2 + \left[\frac{2\operatorname{Im}(z)}{|z|^2+1}\right]^2 + \left[\frac{|z|^2-1}{|z|^2+1}\right]^2 = 1 \right\}$$

(Last eqn is always true: if
$z = u + iv$
$$\frac{4u^2 + 4v^2 + (u^2+v^2-1)^2}{(u^2+v^2+1)^2} = 1.)$$

$$\phi(P \cap S^2) = \{z : 2\,\mathrm{Re}(z) + 2\,\mathrm{Im}(z) = |z|^2 - 1\}$$

Let $z = x + iy$

$$= \{x+iy : 2x + 2y = x^2 + y^2 - 1\}$$

$$= \{x+iy : x^2 - 2x + 1 + y^2 - 2y + 1 = 3\}$$

$$= \{x+iy : (x-1)^2 + (y-1)^2 = 3\}$$ = inverse image of $\{3\}$ under $(x,y) \mapsto (x-1)^2 + (y-1)^2$

= circle of radius $\sqrt{3}$ centred at $(1,1)$.

<u>Example</u>: Algebra/Calc 3 question:
What is the set $x + y = z$ in $\mathbb{R}^3$.

<u>Answer</u>: Let $\psi(x,y,z) = x + y - z \in \mathbb{R}$

$$\psi : \mathbb{R}^3 \to \mathbb{R}.$$

We want $\psi^{-1}(\{0\}) = \{(x,y,z) : \psi(x,y,z) = 0\}$

= plane through $(1,0,1)$, $(0,1,0)$, $(1,-1,0)$.

Note: Inverse Function Thm in Calc 1:

If f is a C^1 fcn on an open (set'd in $\mathbb{C}$) interval $I \subseteq \mathbb{R}$, and if $a \in I$ ($\in U$) and $f'(a) \neq 0$, then

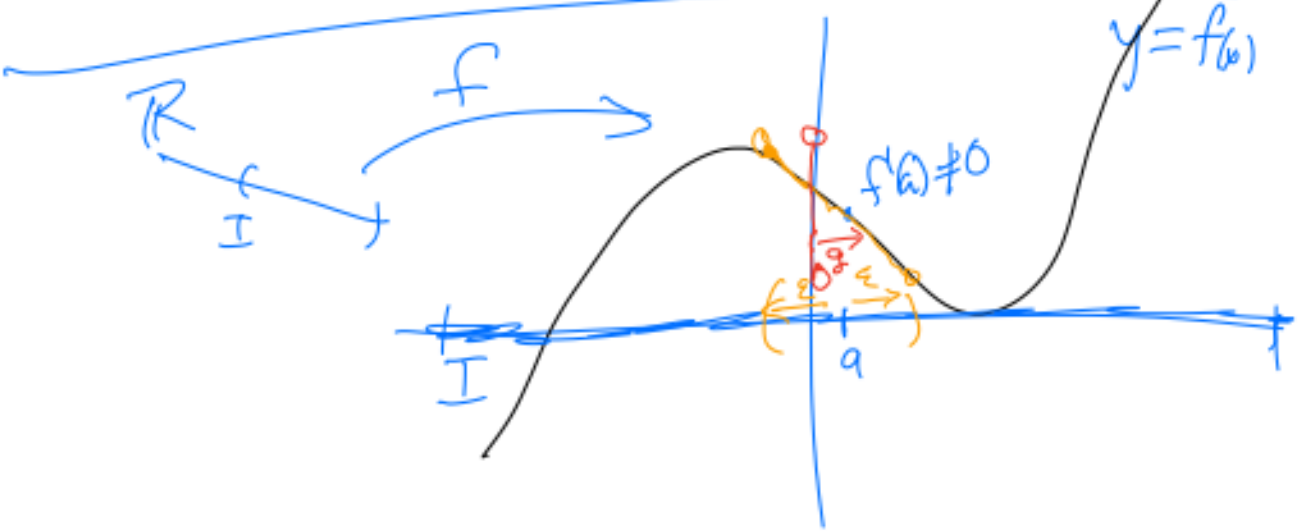
$\exists$ neighborhood $(a-\varepsilon, a+\varepsilon)$ ($B(a,\varepsilon)$) for some $\varepsilon > 0$ and a diff'ble fcn $g : (a-\varepsilon, a+\varepsilon) \to I$ ($B(a,\varepsilon) \to U$) such that

$$f(g(x)) = x \quad \forall x \in (a-\varepsilon, a+\varepsilon)$$

and
$$g(f(x)) = x \quad \forall x \in f((a-\varepsilon, a+\varepsilon))$$

and
$$g'(x) = \frac{1}{f'(g(x))}$$

from: $f(g(x)) = x$
$f'(g(x)) \cdot g'(x) = 1$



Same fact is true in $\mathbb{C}$!